SPATIALLY REGULARIZED Q-BALL IMAGING USING SPHERICAL RIDGELETS

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ABSTRACT
In this note, a novel approach to the problem of estimation of orientation distribution functions (ODFs) in q-ball imaging is presented. Rather than recovering the ODFs in a sequential manner, the proposed method performs a concurrent estimation of a set of ODFs pertaining to a specified volume. In this way, the method takes into account the spatial dependencies between different ODFs which are related to the same neural fiber tracts. Such spatial regularization is proven to be a useful tool to use for low SNR data, in which case it allows substantially improving the directional resolution of q-ball imaging. Finally, the reconstruction is based on representing the ODFs using the multi-resolutional basis of spherical ridgelets, which offers a number of additional advantages such as a significant reduction in data dimensionality and remarkable robustness to noises.

1. INTRODUCTION
The current demand for accurate methods of early diagnosis and assessment of various cerebral disorders necessitates the development of new tools for noninvasive brain imaging. Among such techniques, the advent of diffusion magnetic resonance imaging (dMRI) is currently believed to have opened a new epoch in imaging-based diagnosis of brain-related diseases [1]. In particular, the ability of dMRI to resolve the spatial pattern of water diffusion in the brain lies in the basis of recovering the architecture of neural tracts by means of fiber tractography.

A quantum leap in the diagnostic value of dMRI has become possible with the advent of q-ball imaging [2], which allows recovering the ODFs of local diffusion flows directly from the diffusion data collected using high angular resolution diffusion imaging (HARDI). To reduce the adverse effect of measurement noises, most of the current methods of q-ball imaging precede the computation of ODFs by projecting the HARDI data onto a properly defined signal space [3]. The latter can be specified in terms of, e.g., spherical harmonics [4] or spherical ridgelets [5].

Unfortunately, the directional resolution of q-ball imaging (as controlled by the so-called b-factor) is known to be impossible to increase without intensifying the measurement noise. In this case, to improve the accuracy of q-ball imaging, additional constraints need to be imposed on estimation of ODFs. A possible way to define such constraints is proposed in the present note. Specifically, given a set of HARDI signals acquired within a volume-of-interest, the proposed method recovers their associated ODFs simultaneously using a computationally efficient scheme known as iterative shrinkage [6]. The latter is derived under two basic assumptions, viz.: 1) HARDI signals are sparsely representable in the multi-resolutional basis of spherical ridgelets [5]; 2) the ODFs related to the same diffusion flows are spatially correlated. The experimental study reported in his note shows that the proposed method is capable of substantially improving the accuracy of estimation of ODFs in low SNR settings.

2. Q-BALL IMAGING
Given a HARDI signal \( S(\mathbf{u}) \) (with \( \mathbf{u} \in \mathbb{S}^2 \)), its related ODF can be closely approximated using the Funk-Radon transform (FRT) \( \mathcal{R} \) as given by [2]

\[
\psi(\mathbf{u}) \approx \mathcal{R}\{S\}(\mathbf{u}) = \int_{\mathbb{S}^2} \delta(\mathbf{u} \cdot \mathbf{v}) S(\mathbf{v}) \, d\eta(\mathbf{v}),
\]

where \( \delta \) denotes the delta function, the dot stands for the Euclidean inner product in \( \mathbb{R}^3 \), and \( d\eta \) denotes the standard rotation-invariant measure on \( \mathbb{S}^2 \).

Most of the current methods of q-ball imaging recover the ODFs based on a two-stage processing. At the first stage, the HARDI signals are fitted using a model of the type [3]

\[
S(\mathbf{u}) \simeq A(\mathbf{u}) = \sum_{k \in \mathcal{I}} c_k \varphi_k(\mathbf{u}),
\]

with \( \{\varphi_k\}_{k \in \mathcal{I}} \) being a set of basis functions of some nature. At the second stage, the approximation \( A(\mathbf{u}) \) is subjected to the FRT according to (1) to result in an estimate of \( \psi \).

It is important to note that computing the approximation \( A(\mathbf{u}) \) in (2) serves two important purposes. First, it effectively

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filters the data signals, thereby rendering the subsequent analysis robust to noise. Second, it facilitates the computation of corresponding ODFs, as there are many useful definitions of \( \{ \varphi_k \}_{k \in \mathbb{Z}} \) which admit closed-form expressions for their FRT.

### 3. Ridgelets-Based q-Ball Imaging

#### 3.1. The morphology of HARDI signals

Under some general assumptions [3], the diffusion-weighted signal \( S \) originating from a voxel supporting \( M \) neural fibers can be modeled as a superposition of \( M \) unimodal Gaussian signals of the form \( S_k(u) := \exp\{-b \{ u^T D_k u \} \} \), where \( b \) is a data acquisition parameter (defined by the strength and form of the diffusion-encoding gradients) and \( D_k \) is the diffusion tensor associated with the \( k \)-th fiber. For the case of \( M = 3 \), such elementary signals are exemplified in Fig. 1, which show \( \{ S_k \}_{k=1}^M \) along with their related \( S \) and ODF \( \psi \). Analyzing the shapes of \( S_k \) leads to an important conclusion: the energy of \( S_k \) is localized alongside the great circles whose poles coincide with the diffusion directions associated with \( S_k \). Thus, to maximize the effectiveness of approximation (2), it is imperative to use \( \varphi_k \) whose energy is also compactly supported alongside the great circles. One of the possible definitions of such \( \varphi_k \) has been recently proposed in [5], in which case they are referred to as spherical ridgelets.

#### 3.2. Spherical ridgelets

Spherical ridgelets are constructed using the fundamental principles of wavelet theory. Let \( \kappa_\rho(x) = \exp\{-\rho x (x+1)\} \), where \( 0 < \rho < 1 \) and \( x > 0 \). Also, let \( \kappa_j(x) = \kappa_\rho(2^{-j} x) \) be a scaled version of \( \kappa_\rho \), where \( j \in \mathbb{N} \). Then, one can define a multiresolution set of Gaussian-Weierstrass scaling functions as

\[
K_j(\cdot, v) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \kappa_j(n) P_n(\cdot, v),
\]

where \( P_n \) denotes the Legendre polynomial of order \( n \), \( v \) defines the position of \( K_j \) on \( S^2 \), and \( j = 0, 1, \ldots \) controls the resolution of the corresponding \( K_j \). Consequently, following [5], the semi-discrete frame \( U \) of spherical ridgelets can be defined as

\[
U := \{ \Psi_j(\cdot, v) \mid v \in S^2, j = -1, 0, 1, 2, \ldots \},
\]

where

\[
\Psi_j(\cdot, v) = \begin{cases} (2\pi)^{-1} R \{ K_0(\cdot, v) \}, & \text{if } j = -1, \\
(2\pi)^{-1} R \{ K_{j+1}(\cdot, v) - K_j(\cdot, v) \}, & \text{if } j \in \mathbb{N}
\end{cases}
\]

where \( R \) stands for the FRT, as before. The set \( U \) is dense in the subspace of symmetric HARDI signals, and therefore any \( S \) can be represented in terms of \( \Psi_j \). Moreover, spherical ridgelets have been designed to have their energy compactly supported alongside the great circles, which makes them highly correlated with \( S \). As a result, using \( \Psi_j \) leads to sparse representation of HARDI signals, in which case any given \( S \) can be represented by as few as 6 spherical ridgelets with the precision exceeding that of representing \( S \) with 45 spherical harmonics [5].

Finally, it should be noted that, given an approximation \( A \) of \( S \) in terms of spherical ridgelets, the corresponding ODF \( \psi \) can be straightforwardly computed according to

\[
\psi = \sum_{i=1}^{L} c_i R \{ \Psi_{j_i}(\cdot, v_i) \},
\]

where \( \{ \Psi_{j_i}(\cdot, v_i) \}_{i=1}^{L} \) denotes the set of spherical ridgelets by means of which \( S \) has been approximated.

#### 3.3. Sparse ridgelet analysis via iterative shrinkage

The set \( U \) given by (4) is overcomplete, and therefore finding a unique approximation \( A \) requires using additional constraints. In [5], the uniqueness was achieved via requiring the representation coefficients to be as sparse as possible. Consequently, the orthogonal matching pursuit (OMP) algorithm was used to find the representation coefficients \( \{ c_i \}_{i=1}^{L} \) in (6) along with their associated spherical ridgelets.

Unfortunately, the absence of strong theoretical guarantees attached to OMP as well as the need to predetermine an optimal number of representation coefficients \( L \) limit the practical value of OMP. In this regard, a more robust approximation of HARDI signals is possible by using the tools of convex optimization. To this end, let \( \mathcal{V}_N = \{ v_k \}_{k=1}^{N} \) be a discrete subset of points on \( S^2 \). Also, let \( J > 0 \) be a maximum resolution level defining the finest scale of details in \( S \). Then, a discrete counterpart of \( U \) in (4) can be defined as

\[
\mathcal{U}_{N,J} := \{ \Psi_j(\cdot, v_k) \mid k = 1, \ldots, N, j = -1, \ldots, J \}.
\]
Note that, as opposed to $U$, the set $\cup_{N,J}$ is finite, with its total number of elements equal to $N(J+2)$.

For the sake of notational convenience, let $I$ be the set of paired indices $(k,j)$, with the ranges of $k$ and $j$ defined in (7). Consequently, for any square-summable sequence of coefficients $c = \{c_i\}_{i \in I} \in \ell_2(I)$, one can define

$$\Psi : \ell_2(I) \rightarrow \mathbb{L}_2(S^2) : c \mapsto \sum_{i \in I} c_i \Psi_i,$$

(8)
to be a synthesis operator which maps $c = \{c_i\}_{i \in I}$ to a vector in $\text{span}\{\Psi_i\}_{i \in I}$. Consequently, given a noisy measurement $\hat{S}$ of the original HARDI signal $S$, the latter can be approximated as

$$S \approx \sum_{i \in I} c_i^* \Psi_i,$$

(9)
with

$$c^* = \text{argmin}_c \left\{ \frac{1}{2} \left\| \Psi(c) - \hat{S} \right\|_2^2 + \lambda \left\| c \right\|_1 \right\},$$

(10)
where $\| \cdot \|_2$ and $\| \cdot \|_1$ stand for the $\mathbb{L}_2$- and $\ell_1$-norm, respectively, while $\lambda > 0$ is a regularization constant proportional to the level of measurement noises. It should be noted that the first term of the cost functional in (10) implies that the noises are Gaussian in nature, which is, of course, an approximation.

Minimizing the $\ell_1$-norm of $c$ in (10) forces the solution to be sparse, in which case the majority of coefficients $c_i^*$ are equal to zero. Such a solution can be efficiently computed by means of iterative shrinkage [6]. Specifically, let $\Psi^*$ denote the adjoint of $\Psi$ defined as

$$\Psi^* : \mathbb{L}_2(S^2) \rightarrow \ell_2(I) : F \mapsto \{\langle F, \Psi_i \rangle\}_{i \in I},$$

(11)
where $\langle \cdot, \cdot \rangle$ denotes the standard inner product, i.e. $\langle F, \Psi_i \rangle = \int_{S^2} F(u) \Psi_i(u) d\eta(u)$. Consequently, given a $\mu > \|\Psi\Psi^*\|$, the solution of (10) can be computed as a stationary point of the sequence of estimates produced by

$$c^{t+1} = \mathcal{S}_\tau \{c^t + \mu^{-1} \Psi^* \{\hat{S} - \Psi \{c^t\}\}\},$$

(12)
where $\mathcal{S}_\tau$ stands for the operation of soft-thresholding, i.e. $\mathcal{S}_\tau : x \mapsto \text{sign}(x)(|x| - \tau)_+$. Finally, we note that the computation of an optimal $c^*$ by means of (12) requires only the application of operators $\Psi$ and $\Psi^*$, which can be done by means of closed-form expressions as detailed in [5].

4. SPATIAL REGULARIZATION

The $b$-factor in the definition of $S_k$ in Section 3.1 controls the directional resolution of q-ball imaging, with higher values of $b$ resulting in better directionality of the resulting ODFs. Unfortunately, using relatively large values of $b$ is usually avoided in practice, as it inevitably reduces the SNR. In such a case, it is therefore necessary to find additional means of improving the accuracy of q-ball imaging. One of the possible solutions to the above problem is to exploit the spatial continuity of neural fiber tracts, which suggests that neighboring ODFs should exhibit a certain degree of correlations with respect to each other.

To incorporate the spatial smoothness constraints into the problem of estimation of ODFs, the problem domain should be appropriately modified. Specifically, let $\Omega_K = \{z_n \in \mathbb{R}^3 | n = 1, 2, \ldots, K\}$ be a set of $K$ spatial positions uniformly sampled within a predefined volume-of-interest. As well, let $\{S(z_n)\}_{n=1}^K$ be the HARDI signals measured at the points of $\Omega_K$. Then, the optimal ridgelet coefficients $\{c(z_n)\}_{n=1}^K$ (corresponding to $\{S(z_n)\}_{n=1}^K$) can be found via solution of the following optimization problem

$$\min_{\{c(z_n)\}_{n=1}^K} \left\{ \sum_{z_n \in \Omega_K} \left( \frac{1}{2} \|\Psi(c(z_n)) - S(z_n)\|_2^2 + \lambda \|c(z_n)\|_1 + \right) + \mu \sum_{\xi \in \mathcal{N}(z_n)} \omega(\xi, z_n) \|\Psi(z_n) - \Psi(\xi)\|_2^2 \right\},$$

(13)
where $\mu > 0$ is a spatial regularization parameter and $\mathcal{N}(z_n)$ denotes a $3 \times 3 \times 3$ neighborhood of voxels centered around.
zn. Note that the last term in (13) accounts for the spatial smoothness of the recovered ODFs, in which case the weights
0 < ω(ξ, zn) < 1 are supposed to control the degree of correlation of the ODFs located at spatial positions zn and ξ ∈ N(zn). Specifically, in the current work, the weights have been defined to be

ω(ξ, zn) = \frac{(S(ξ), S(zn))}{||S(ξ)||_2 ||S(zn)||_2}.
(14)

The spatial regularization functional in the minimization problem (13) is obviously quadratic, which suggests the possibility to redefine (13) in a more standardized way. In particular, let the ridgelet coefficients \{c(zn)\}_n be combined together in a vector c ∈ R^{K×N(J+2)}. In this case, it is straightforward to show that there exist a linear operator A : R^{K×N(J+2)} → R^{K×N(J+2)}, and a fixed (data-dependent) vector b ∈ R^{K×N(J+2)} such that the problem (13) can be alternatively rewritten as

c^* = \arg\min_c \left\{ \frac{1}{2} ||A(c) - b||_2^2 + \lambda ||c||_1 \right\}
(15)

where c* stands for the optimal solution. Moreover, following [6], c* can be computed by means of an iterative shrinkage procedure analogous to (12). It should be noticed that A in (15) needs neither to be stored nor manipulated as a matrix (whose size would be impractically large even for relatively small values of K). Instead, to find c* using an IS algorithm requires only recursively applying A and its adjoint A*, which are defined through Ψ and Ψ*, respectively. The operators Ψ and Ψ*, in turn, can be efficiently computed by means of closed form expressions as detailed in [5].

5. RESULTS

To test the performance of the proposed method under controllable conditions, HARDI data acquired from a specially designed dMRI phantom [7] was used. Specifically, the data set used in this study was acquired using 3 mm isotropic acquisition over a square field-of-view of size 19.2 cm, b = 2000 s/mm², and 64 diffusion orientations, uniformly distributed over the sphere. (For more details on the data acquisition setup see www.inao.fr/spip.php?article107.)

Subplot A of Fig. (2) shows a b₀-image of the phantom, with the yellow rectangle designating a subset of the image domain over which a closer analysis was performed as shown by Subplots B1 and B2 of the same figure. In particular, Subplot B1 shows the ODFs reconstructed by solving (13) with λ = 0.2 and μ = 0 (i.e. without spatial regularization). Note that, due to the sparseness of ridgelet representation, most of the estimated coefficients were found to be equal to zero. For this reason, the ODFs in Fig. 2 were recovered using only 6 largest ridgelet representation coefficients per estimate. At the same time, Subplot B2 of the figure shows the ODFs recovered using spatial regularization (μ = 0.1), with the rest of estimation parameters left intact. One can see that, in this case, the field of recovered ODFs appears to be much more regularized as compared to the previous case. This difference can be clearly seen in Subplots C1 and C2 which show the ODFs corresponding to the spatial locations indicated by the yellow rectangles in Subplots B1 and B2, respectively. Note that these locations correspond to the region of crossing of two fiber tracts, and, therefore, the associated ODFs should have two-modal structures. This structure, however, can only be observed in the regularized estimates (Subplot C2), which proofs the superiority of the proposed method over the case when no spatial regularization is used.

Finally, to quantify the improvement in accuracy of reconstruction of ODFs, two performance metrics were computed and compared. Specifically, in the non-regularized case, the directional error of estimated ODFs was found to be equal to 5.62°, while in the regularized case, the error was found to be equal to 3.17°. Additionally, the rate of correct detection of a two-fiber diffusion pattern was found to be equal to 48.1% and 93.5% in the cases of non-regularized and regularized estimation, respectively.

The above results clearly demonstrate that augmenting the estimation of ODFs by incorporating the spatial smoothness constraints can result in a substantial improvement in the accuracy of q-ball imaging.

6. REFERENCES